MST121 EBB

The Open University

USING MATHEMATICS

Exercise Booklet B

Exercise Booklet

B

SUP 60835 7
1.1

Contents

The exercises in this booklet are intended to give further practice, should you require it, in handling the main mathematical ideas in each chapter of MST121, Block B. The exercises are ordered by chapter and section, and are numbered correspondingly: for example, Exercise 3.2 for Chapter B1 is the second exercise on Section 3 of that chapter.

Exercises for Chapter B1

Section 1

Exercise 1.1

(a) Use sigma notation to express the sum

$$3^2 + 3^3 + \cdots + 3^{12}.$$

(b) What is the value of this sum?

Exercise 1.2

Find the values of each of the following.

(a) $\displaystyle\sum_{i=3}^{5} i^2$

(b) $\displaystyle\sum_{i=1}^{30} i$

(c) $\displaystyle\sum_{i=19}^{42} i$

(d) $\displaystyle\sum_{i=1}^{40} (\tfrac{1}{5} - \tfrac{1}{2}i)$

Exercise 1.3

(a) Find the sum of the integers from 40 to 89, inclusive.

(b) Hence find the sum of the numbers

$$3.4,\ 3.41,\ 3.42, \ldots,\ 3.88,\ 3.89.$$

Exercise 1.4

(a) (i) Any even integer can be written in the form $2i$, where i is an integer. Use this fact to express in sigma notation the sum of the first 100 positive even integers, that is,

$$2 + 4 + 6 + \cdots + 198 + 200.$$

(ii) Find the sum of the first 100 positive even integers.

(b) Find the sum of the first 200 positive odd integers, that is,

$$1 + 3 + 5 + \cdots + 397 + 399.$$

Section 2

Exercise 2.1

Suppose that P_n is the human population of a country at n years after 1 January 2000. Suppose that P_0, the population on 1 January 2000, is 30 million. Assume that the population alters only through births and deaths, rather than (for example) through migration.

(a) Suppose that the number of births in each subsequent year (from 1 January to 31 December) is 48 per thousand of the population at the start of the year and that the number of deaths is 39 per thousand.

(i) Find a recurrence system that P_n must satisfy.

(ii) Write down a closed form for P_n.

(iii) According to this model, what is the population on 1 January 2010?

(iv) What form of population variation does the model predict in the long term?

(b) Suppose that the number of births in each subsequent year (from 1 January to 31 December) is 33 per thousand of the population at the start of the year and that the number of deaths is 37 per thousand.

(i) Find the annual proportionate growth rate of the population.

(ii) Write down a closed form for P_n.

(iii) According to this model, what is the population on 1 January 2020?

(iv) What form of population variation does the model predict in the long term?

Exercise 2.2

Late in 1994, it was estimated that the world human population at mid-year 1994 was 5607 million, representing a 90 million increase since mid-year 1993 and a 600 million increase since mid-year 1987.

(a) Assuming that this population can be modelled exactly between mid-year 1987 and mid-year 1994 by an exponential model, find to 3 significant figures the annual proportionate growth rate for this period.

(b) How does the population estimated by this model for mid-year 1993 compare with the actual figure?

(c) At which mid-year does the model predict that the world human population will first have exceeded 6000 million (or, equivalently, 6 billion)?

Section 3

Exercise 3.1

Observations of the birth and death rates of foxes on an island suggest the following description of the behaviour of the population:

◇ the annual proportionate death rate is constant at 0.22;

◇ the annual proportionate birth rate decreases linearly with the population size P, according to the formula $0.47 - 2 \times 10^{-4}P$.

The population on 1 January 1999 was 400.

(a) Find a recurrence system for P_n, the population of foxes n years after 1 January 1999.

(b) Show that the recurrence relation obtained is logistic, by identifying the values of the parameters r and E from the logistic recurrence relation which apply in this case.

(c) What population does the model predict for 1 January 2001?

Exercise 3.2

In a series of experiments, different numbers of rats are introduced into similar controlled environments, and the numbers in the next generation counted. When the experiment is started with 80 rats, the rat population three months later is 288, whereas when the experiment is started with 200 rats, there are 120 after three months.

(a) Write down the proportionate growth rate per three months, $(P_{n+1} - P_n)/P_n$, for each of $P_n = 80$ and $P_n = 200$.

(b) Assuming that the behaviour of this population satisfies the logistic model, estimate the values of the equilibrium population level, E, and the proportionate growth rate for low populations, r.

(c) If the experiment is started with 100 rats, how many will there be in three months time, according to the model?

Exercise 3.3

A UN analysis in 1998 suggested that the proportionate growth rate per year of the world population was 0.013 when the population size was 5.66 billion. Consider a scenario based on this data and on the logistic model, where it is also assumed that the proportionate growth rate per year is 0.006 for a world population size of 6.36 billion.

(a) Estimate the values of the equilibrium population level, E, and the parameter r used for this application of the logistic model.

(b) Using this model, and the world population estimate of 6.06 billion for 2000, what population is forecast for the year 2002?

Section 5

Exercise 5.1

Consider the sequences defined by the logistic recurrence relation

$$P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right)$$

with the parameters stated below. Basing your answers on the investigations in Chapter B1 Section 4, decide in each case whether or not the sequence converges and, if it does, to what limit.

(a) $r = 2.5$, $E = 800$, $P_0 = 97$

(b) $r = 1.3$, $E = 811$, $P_0 = 500$

Exercise 5.2

The following sequences are given by recurrence systems. In each case, find all possible limit values for the sequence. Then use methods from Chapter A1 to find a closed form for the sequence, and use this to determine by reasoning the long-term behaviour of the sequence.

(a) $x_0 = 5, \quad x_{n+1} = -0.2x_n + 12 \quad (n = 0, 1, 2, \ldots)$

(b) $x_0 = 8, \quad x_{n+1} = 2.4x_n - 7 \quad (n = 0, 1, 2, \ldots)$

(c) $x_0 = 5, \quad x_{n+1} = 2.4x_n - 7 \quad (n = 0, 1, 2, \ldots)$

Exercise 5.3

The following sequences are given by closed-form formulas. In each case, use reasoning to decide whether or not the sequence converges and, if it does, to what limit. In each case $n = 1, 2, 3, \ldots$.

(a) $a_n = 1 - 2^n$ (b) $a_n = 1 + 1/n^2$

(c) $a_n = \dfrac{1}{1 + 2^n}$ (d) $a_n = n - 1/n$

(e) $a_n = 6(-0.9)^n$ (f) $a_n = \dfrac{2n - 3}{7 - 5n}$

(g) $a_n = \dfrac{n^2 + 1}{3n^2 - 5}$ (h) $a_n = \dfrac{n^2 + 1}{n + 1}$

(i) $a_n = \dfrac{3^n}{2 + 3^n}$

Exercise 5.4

For each of the following infinite series, find its sum.

(a) $1 + \frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + \cdots$

(b) $\frac{3}{4} - (\frac{3}{4})^2 + (\frac{3}{4})^3 - (\frac{3}{4})^4 + \cdots$

(c) $(\frac{1}{2})^2 + (\frac{1}{2})^3 + (\frac{1}{2})^4 + (\frac{1}{2})^5 + \cdots$

Exercise 5.5

For each of the following recurring decimals, find the fraction equivalent to it.

(a) $0.222\,222\ldots$ (b) $0.416\,241\,624\,162\ldots$

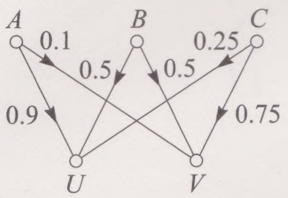
(c) $0.317\,171\,717\ldots$ (d) $0.198\,019\,801\,980\ldots$

Exercises for Chapter B2

Section 1

Exercise 1.1

The following network diagram represents water flow from inputs at nodes A, B, C to outputs at nodes U, V.



(a) How much water is output at node V if 5 litres of water are input at node A?

(b) How much water is output at nodes U and V if 2 litres of water are input at each of nodes A and C and 1 litre is input at node B?

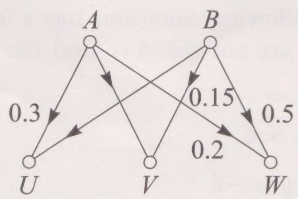
(c) How much water is output at nodes U and V if x litres of water are input at node A, y litres at node B and z litres at node C?

(d) Write down a vector which represents the input in part (b).

(e) Write down the matrix which represents this network.

Exercise 1.2

The following network diagram represents water flow from inputs at nodes A, B to outputs at nodes U, V, W.



(a) What are the missing numbers for the pipes from A to V and from B to U?

(b) Write down the matrix which represents this network.

(c) Suppose that x litres of water are input at node A and y litres at node B, and that the outputs are u litres at node U, v litres at node V and w litres at node W. Write down a matrix equation that connects the corresponding input and output vectors.

Exercise 1.3

(a) Find the output vectors when the following input vectors are acted on by the matrix

$$\begin{pmatrix} 0.1 & 0.3 & 0.5 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.7 & 0.1 \end{pmatrix}.$$

(i) $\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 100 \\ 20 \\ 60 \end{pmatrix}$

(b) Sketch the network corresponding to the matrix in part (a), and add labels to show the proportions of water flowing through each pipe.

Section 2

Exercise 2.1

Evaluate each of the following.

(a) $\begin{pmatrix} 7 & 10 & -3 \\ -4 & 7 & 11 \end{pmatrix} + \begin{pmatrix} 14 & 10 & 5 \\ 4 & -20 & -6 \end{pmatrix}$

(b) $\begin{pmatrix} 7 & 10 & -3 \\ -4 & 7 & 11 \end{pmatrix} - \begin{pmatrix} 14 & 10 & 5 \\ 4 & -20 & -6 \end{pmatrix}$

(c) $-4\begin{pmatrix} 8 & \frac{1}{2} \\ -2 & 1 \end{pmatrix}$

(d) $\frac{2}{5}\begin{pmatrix} 7 & -15 \\ -2 & 4 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & -1 & -2 \\ -8 & 1 & 6 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 4 & 0 \\ 6 & -2 \end{pmatrix}$

(f) $\begin{pmatrix} 6 & 5 & 0 & 2 \\ -1 & 0 & 9 & -2 \\ 1 & 6 & 3 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ -2 \\ 4 \end{pmatrix}$

Exercise 2.2

This exercise involves the following matrices.

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 2 & -4 & 7 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 4 & 2 \\ -6 & 0 \\ 3 & 1 \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} 6 & 0 \\ -2 & 1 \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} -6 & -5 & 3 \\ -1 & 5 & -3 \end{pmatrix}$$

(a) Write down the following elements of these matrices:

(i) a_{23} (ii) d_{21} (iii) f_{12}

(b) Which of the following sums can be formed? For each which can be formed, evaluate the sum.

$\mathbf{A}+\mathbf{B}$ $\mathbf{A}+\mathbf{F}$ $\mathbf{B}+\mathbf{B}$

$\mathbf{B}+\mathbf{D}$ $\mathbf{B}+\mathbf{E}$ $\mathbf{F}-\mathbf{A}$

(c) Which of the following products can be formed? For each which can be formed, evaluate the product.

$\mathbf{AB}$ $\mathbf{BF}$ $\mathbf{BE}$ $\mathbf{EB}$ $\mathbf{FD}$

$\mathbf{DA}$ $\mathbf{D}^2$ $\mathbf{B}^3$ $(\mathbf{CD})\mathbf{E}$ $\mathbf{C}(\mathbf{DE})$

Exercise 2.3

(a) The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 2 \\ 8 & -7 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}.$$

Show that $\mathbf{AC}+\mathbf{BC} = (\mathbf{A}+\mathbf{B})\mathbf{C}$.

(b) The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Show that $\mathbf{AC}+\mathbf{BC} = (\mathbf{A}+\mathbf{B})\mathbf{C}$.

Section 3

Exercise 3.1

The table below gives the age-dependent birth and death rates for two subpopulations (given in millions) of a country at 1 January 2000. The population is subdivided into adults (aged 18 or over) and minors (aged under 18).

	Minors (aged < 18)	Adults (aged ≥ 18)
Subpopulations	3.2	13.5
Birth rate	0.003	0.041
Death rate	0.011	0.019

(a) Let M_n and A_n be the numbers of minors and adults respectively at the start of year n, where 1 January 2000 is taken as the start of year 0.

Draw a network diagram showing the interrelationships between the subpopulations M_0, A_0 and M_1, A_1. (Assume that by the end of each year, $\frac{1}{18}$ of the surviving subpopulation of minors have become adults.)

(b) Write down the corresponding matrix model for the population.

(c) Find the total population predicted by the model for 1 January 2002.

Section 5

Exercise 5.1

For each of the following matrices, write down its inverse or show that the inverse does not exist.

(a) $\mathbf{A} = \begin{pmatrix} 7 & 2 \\ 6 & 2 \end{pmatrix}$

(b) $\mathbf{B} = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} \frac{1}{3} & \frac{1}{5} \\ 5 & -\frac{1}{2} \end{pmatrix}$

Exercise 5.2

This exercise concerns the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

(a) Find the determinant of the matrix $\mathbf{A}$.

(b) For which values of a and b does $\mathbf{A}$ have an inverse? For these values, write down the inverse matrix.

Exercise 5.3

For each of the following pairs of simultaneous linear equations, rewrite the equations in matrix form and use matrix methods to solve the equations.

(a) $4x + y = 2$
$3x + y = 5$

(b) $-2x + 5y = 8$
$3x + 7y = -2$

(c) $\frac{1}{3}x + 3y = -8$
$\frac{1}{2}x - \frac{3}{2}y = 6$

Exercise 5.4

Which of the following equations has a unique solution? (You are not asked to find the solution.)

(a) $7x + 3y = -1$
$-3x + y = 6$

(b) $-4x + 6y = -6$
$6x - 9y = 5$

Exercises for Chapter B3

Section 1

Exercise 1.1

Let $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = -7\mathbf{i}$ and $\mathbf{c} = \mathbf{i} + 3\mathbf{j}$.

(a) Write down the column form for each of these vectors.

(b) Find the component form of $2\mathbf{a} - 3\mathbf{b} + 4\mathbf{c}$, and specify the **i**-component and **j**-component of this vector.

Exercise 1.2

Let $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$. Sketch arrows to represent each of the vectors

$$\mathbf{a}, \quad \mathbf{b}, \quad -\mathbf{b}, \quad 2\mathbf{b}, \quad \mathbf{a}+\mathbf{b}, \quad \mathbf{a}-\mathbf{b}.$$

Section 2

Exercise 2.1

A vector **a** has magnitude $|\mathbf{a}| = 12$ and direction $\theta = 100°$. Calculate the component form of **a**, giving the components correct to two decimal places.

Exercise 2.2

For each case below, calculate the component form of the vector **a** whose magnitude $|\mathbf{a}|$ and direction θ are given, specifying the components as exact values.

(a) $|\mathbf{a}| = 2, \ \theta = -30°$

(b) $|\mathbf{a}| = 3, \ \theta = 135°$

Exercise 2.3

Find the magnitude and direction of each of the vectors $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$, $\mathbf{b} = 7\mathbf{j}$, $\mathbf{c} = 5\mathbf{i} + 2\mathbf{j}$, and of the vector $\mathbf{a} - \mathbf{c}$, giving your answers correct to one decimal place.

Exercise 2.4

Suppose that **i** is 1 km East and **j** is 1 km North.

(a) Find the direction θ that corresponds to each of the following bearings.

(i) N 20°W (ii) S 72°E

(b) Find the bearing that corresponds to each of the following directions.

(i) $\theta = 65°$ (ii) $\theta = -165°$

Exercise 2.5

The displacement from Edinburgh to Glasgow is 69 km at S 81°W. The displacement from Edinburgh to Stirling is 50 km at N 68°W. In terms of direct distance and a bearing, find

(a) the displacement from Stirling to Edinburgh;

(b) the displacement from Stirling to Glasgow, giving answers to one decimal place.

Exercise 2.6

A boat has a speed in still water of $4\,\mathrm{m\,s^{-1}}$ and is pointed in the direction S 10°E, but there is a current of speed $2.5\,\mathrm{m\,s^{-1}}$ flowing towards the direction N 75°W. Find the resultant velocity of the boat, in terms of its speed (to one decimal place) and a bearing (to the nearest degree).

Exercise 2.7

A bird has a speed in still air of $15\,\mathrm{m\,s^{-1}}$ and is pointed in the direction S 80°W, but it flies in a wind of speed $8\,\mathrm{m\,s^{-1}}$ blowing from S 5°E. Find the velocity of the bird relative to the ground, in terms of its speed and a bearing. Give your answers to one decimal place.

Section 3

Exercise 3.1

Solve the triangle ABC for which $b = 9$, $A = 24°$ and $C = 53°$, giving the side lengths to one decimal place.

Exercise 3.2

(a) Given the information that a triangle ABC has $b = 14$, $c = 7$ and $C = 20°$, find the possible values of the angle B, giving your answers to one decimal place.

(b) Given the further piece of information that the longest side of the triangle is BC, solve the triangle.

Exercise 3.3

Use the Cosine Rule to solve the triangle ABC for which $a = 6$, $b = 14$ and $c = 11$, giving the angles to two decimal places.

Exercise 3.4

A boat has a speed in still water of $10\,\mathrm{m\,s^{-1}}$ and is pointed South-East, but there is a current of $3\,\mathrm{m\,s^{-1}}$ flowing East. By applying the Cosine Rule or Sine Rule, find the resultant velocity of the boat, in terms of its speed and a bearing. Give your answers to one decimal place.

Section 4

Exercise 4.1

An object which remains at rest is acted upon by three forces $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$, where $\mathbf{P}$ and $\mathbf{R}$ are given in terms of specified Cartesian unit vectors by

$$\mathbf{P} = -5\mathbf{i} + \mathbf{j}, \quad \mathbf{R} = 3\mathbf{i} + 4\mathbf{j}.$$

Find the component form of the force $\mathbf{Q}$.

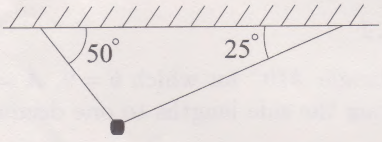
Exercise 4.2

An object which remains at rest is acted upon by three forces $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$. The force $\mathbf{P}$ has magnitude $|\mathbf{P}| = 70\,\mathrm{N}$ and direction $\theta = 80°$. The force $\mathbf{Q}$ has direction $\theta = -45°$ and the force $\mathbf{R}$ has direction $\theta = -160°$. By considering the triangle of forces, find the magnitudes of the forces $\mathbf{Q}$ and $\mathbf{R}$, giving your answers to one decimal place.

Section 5

Exercise 5.1

A hollow barrel of mass 20 kg is hanging from a horizontal ceiling on two cords, as in the diagram below. One cord makes an angle of 50° with the ceiling and the other cord makes an angle of 25° with the ceiling.



Find the magnitudes of the tensions from the cords, by each of the methods below. In each case, take $g = 10\,\mathrm{m\,s^{-2}}$ and give your answers to one decimal place.

(a) Use components.

(b) Use the triangle of forces.

Exercise 5.2

A bronze paperweight, of mass 0.5 kg, is at rest on a smooth flat surface at an angle of 40° to the horizontal.

(a) The paperweight is prevented from sliding down the slope by a string parallel to the surface which pulls up the slope. Using components, and taking $g = 10\,\mathrm{m\,s^{-2}}$, find the magnitudes of the tension from the string and of the normal reaction from the surface, giving your answers to one decimal place. (Choose $\mathbf{i}$ to represent 1 N upwards along the sloping surface and $\mathbf{j}$ to represent 1 N in the direction normal to the surface.)

(b) The paperweight is instead held in position by a string pulling upwards at an angle of 10° to the surface, that is, at an angle of 50° to the horizontal. Use the triangle of forces to find the magnitude of the tension from the string, giving your answer to one decimal place.

Solutions for Chapter B1

Solution 1.1

(a) The sum may be written as

$$3^2 + 3^3 + \cdots + 3^{12} = \sum_{i=2}^{12} 3^i.$$

(b) Using the formula

$$\sum_{i=0}^{n} ar^i = a\left(\frac{1-r^{n+1}}{1-r}\right) \quad (r \neq 1)$$

from Chapter B1 Subsection 1.2, with $a = 3^2$, $r = 3$ and $n = 10$, the value of this sum is

$$\begin{aligned}\sum_{i=0}^{10}(3^2 \times 3^i) &= \frac{3^2(1-3^{11})}{1-3} \\ &= \tfrac{9}{2}(3^{11}-1) \\ &= 797\,157.\end{aligned}$$

Solution 1.2

(a) $\displaystyle\sum_{i=3}^{5} i^2 = 3^2 + 4^2 + 5^2 = 50.$

(b) Using the formula

$$\sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1)$$

from Chapter B1 Subsection 1.2, with $n = 30$, the value of the sum is

$$\sum_{i=1}^{30} i = \tfrac{1}{2} \times 30 \times 31 = 465.$$

(c) Using the same formula as in part (b), with $n = 42$ and with $n = 18$ in turn, the sum is

$$\begin{aligned}\sum_{i=19}^{42} i &= \sum_{i=1}^{42} i - \sum_{i=1}^{18} i \\ &= \tfrac{1}{2} \times 42 \times 43 - \tfrac{1}{2} \times 18 \times 19 \\ &= 732.\end{aligned}$$

(d) Using the formula

$$\sum_{i=1}^{n}(a + bx_i) = an + b\sum_{i=1}^{n} x_i$$

from Chapter B1 Subsection 1.2, with $a = \frac{1}{5}$, $b = -\frac{1}{2}$, $x_i = i$ and $n = 40$, the sum is

$$\begin{aligned}\sum_{i=1}^{40}\left(\tfrac{1}{5} - \tfrac{1}{2}i\right) &= \tfrac{1}{5} \times 40 - \tfrac{1}{2}\sum_{i=1}^{40} i \\ &= 8 - \tfrac{1}{2}\left(\tfrac{1}{2} \times 40 \times 41\right) \\ &= -402.\end{aligned}$$

Solution 1.3

(a) The sum of the integers from 40 to 89, inclusive, is

$$\begin{aligned}\sum_{i=40}^{89} i &= \sum_{i=1}^{89} i - \sum_{i=1}^{39} i. \\ &= \tfrac{1}{2} \times 89 \times 90 - \tfrac{1}{2} \times 39 \times 40 \\ &= 3225.\end{aligned}$$

(b) The numbers given form a finite arithmetic sequence, which can be expressed as

$$3 + 0.01i \quad (i = 40, 41, \ldots, 89).$$

The sum of these terms is

$$\begin{aligned}\sum_{i=40}^{89}(3 + 0.01i) &= 3 \times 50 + 0.01\sum_{i=40}^{89} i \\ &= 150 + 0.01 \times 3225 \quad \text{(from (a))} \\ &= 182.25.\end{aligned}$$

Solution 1.4

(a) (i) The sum of the first 100 positive even integers can be written as $\displaystyle\sum_{i=1}^{100} 2i$.

(ii) The sum is

$$\begin{aligned}\sum_{i=1}^{100} 2i &= 2\sum_{i=1}^{100} i \\ &= 2\left(\tfrac{1}{2} \times 100 \times 101\right) \\ &= 10\,100.\end{aligned}$$

(b) Any odd integer can be written in the form $2i - 1$, where i is an integer. The sum of the first 200 positive odd integers is

$$\begin{aligned}\sum_{i=1}^{200}(2i-1) &= 2\sum_{i=1}^{200} i - 1 \times 200 \\ &= 2\left(\tfrac{1}{2} \times 200 \times 201\right) - 200 \\ &= 40\,000.\end{aligned}$$

(There are many other correct ways of solving this problem.)

Solution 2.1

(a) (i) During the $(n+1)$st year, the 'joiners' will be the people born during the year, of which there are $\frac{48}{1000}P_n = 0.048P_n$; and the 'leavers' will be the deaths, of which there are $0.039P_n$. The difference between births and deaths gives the increase in the population during the year (since we are assuming no migration), so that

$$P_{n+1} - P_n = 0.048P_n - 0.039P_n = 0.009P_n.$$

Thus the sequence P_n, taking P_n in millions, is given by

$$P_0 = 30, \quad P_{n+1} = 1.009P_n \quad (n = 0, 1, 2, \ldots).$$

(ii) This recurrence system describes a geometric sequence with closed form

$$P_n = 30(1.009)^n \quad (n = 0, 1, 2, \ldots).$$

(iii) The population on 1 January 2010, which corresponds to $n = 10$, is given by

$$P_{10} = 30(1.009)^{10} \simeq 32.8,$$

so is approximately 32.8 million.

(iv) This model predicts that the population will increase, and do so more and more rapidly.

(b) (i) During the $(n+1)$st year, the 'joiners' will be the people born during the year, of which there are $0.033P_n$; and the 'leavers' will be the deaths, of which there are $0.037P_n$. The difference between births and deaths gives an increase in the population during the year of $-0.004P_n$ (which, as it is negative, really represents a decrease). The annual proportionate growth rate of the population is the change in the population over the year divided by the population at the start of the year, which equals

$$\frac{-0.004P_n}{P_n} = -0.004.$$

(ii) As the proportionate growth rate is constant and equals -0.004, the exponential model gives the closed form

$$P_n = (1 - 0.004)^n P_0,$$

and putting in the value of $P_0 = 30$ (million), this becomes

$$P_n = 30(0.996)^n.$$

(iii) The population on 1 January 2020, which corresponds to $n = 20$, is given by

$$P_{20} = 30(0.996)^{20} \simeq 27.7,$$

so is approximately 27.7 million.

(iv) This model predicts that the population will decrease, ultimately to extinction.

Solution 2.2

(a) Take $n = 0$ as mid-year 1987, and let P_n be the population n years later in millions. Then $n = 7$ at mid-year 1994. Using the information given, $P_7 = 5607$ and $P_0 = P_7 - 600 = 5007$. According to the exponential model, we have $P_n = (1+r)^n P_0$, so that, in particular,

$$P_7 = (1+r)^7 P_0, \quad \text{that is,}$$
$$5607 = (1+r)^7 \times 5007.$$

On solving this equation for r, we obtain

$$r = \left(\frac{5607}{5007}\right)^{1/7} - 1 = 0.0163 \quad \text{(to 3 s.f.)}.$$

This is just over 1.6% per year.

(b) With this value of r, the model gives the population in 1993 (when $n = 6$) as

$$P_6 = 5007(1+r)^6 \simeq 5517,$$

which is approximately 5517 million. From the information given in the question, the actual population in 1993 was $5607 - 90 = 5517$ (million). Thus the model produces a very accurate estimate for the population in 1993.

(c) We need to find the smallest value of n for which the value of P_n obtained from the model exceeds 6000. Using the figure 0.0163 for r, one way of doing this is to solve for n the equation

$$5007(1.0163)^n = 6000,$$

or equivalently,

$$(1.0163)^n = \frac{6000}{5007}.$$

Using the methods of Chapter A3 Subsection 4.3, we apply the function ln to both sides of this equation to obtain

$$n \ln(1.0163) = \ln\left(\frac{6000}{5007}\right).$$

It follows that

$$n = \frac{1}{\ln(1.0163)} \times \ln\left(\frac{6000}{5007}\right) \simeq 11.19.$$

Thus the smallest integer value of n for which the population exceeds 6000 million is 12. This corresponds to mid-year 1999.

Solution 3.1

(a) The proportionate growth rate $R(P)$ at population P is births minus deaths, that is,

$$\begin{aligned} R(P) &= (0.47 - 0.0002P) - 0.22 \\ &= 0.25 - 0.0002P. \end{aligned}$$

Hence the actual growth for the year which starts at time n is $(0.25 - 0.0002P_n)P_n$. Since this is also the increase from P_n to P_{n+1}, we have the recurrence system

$$P_0 = 400, \quad P_{n+1} - P_n = (0.25 - 0.0002P_n)P_n.$$

(b) The right-hand side of this recurrence relation can be written as

$$0.25P_n\left(1 - \frac{0.0002}{0.25}P_n\right),$$

which is of the logistic form $rP_n(1 - P_n/E)$ with $r = 0.25$ and $E = 0.25/0.0002 = 1250$.

(c) The population predicted for 1 January 2001 is given by P_2. Using the recurrence relation obtained in part (a), we have

$$\begin{aligned} P_1 - P_0 &= (0.25 - 0.0002P_0)P_0 \\ &= (0.25 - 0.0002 \times 400)400 \\ &= 68, \end{aligned}$$

so that $P_1 = P_0 + 68 = 468$. Then

$$\begin{aligned} P_2 - P_1 &= (0.25 - 0.0002P_1)P_1 \\ &= (0.25 - 0.0002 \times 468)468 \\ &= 73.1952, \end{aligned}$$

so that, rounding this down to the integer 73, the predicted population is $P_2 = P_1 + 73 = 541$.

Solution 3.2

(a) The proportionate growth rate is

$$\frac{288 - 80}{80} = 2.6 \quad \text{when } P_n = 80, \quad \text{and}$$
$$\frac{120 - 200}{200} = -0.4 \quad \text{when } P_n = 200.$$

(b) Since the proportionate growth rate has the form $r(1 - P/E)$ for a population P (equation (3.2) of Section 3), we have

$$2.6 = r\left(1 - \frac{80}{E}\right), \quad -0.4 = r\left(1 - \frac{200}{E}\right).$$

On dividing through each equation by r and then rearranging, we obtain a pair of simultaneous linear equations in $1/r$ and $1/E$:

$$\frac{2.6}{r} + \frac{80}{E} = 1 \quad \text{and} \quad -\frac{0.4}{r} + \frac{200}{E} = 1.$$

Eliminating the $1/E$ term, by subtracting two times the second equation from five times the first equation, gives

$$\frac{13.8}{r} = 3, \quad \text{that is,} \quad r = 4.6.$$

On substituting this value for r into the first equation, we obtain

$$\frac{2.6}{4.6} + \frac{80}{E} = 1, \text{ that is, } \frac{80}{E} = 1 - \frac{2.6}{4.6} = \frac{2}{4.6}.$$

Solving this equation gives $E = 184$.

(c) The logistic recurrence relation in this case is

$$P_{n+1} - P_n = 4.6P_n\left(1 - \frac{P_n}{184}\right).$$

If $P_n = 100$, then we have

$$P_{n+1} = 100 + 4.6 \times 100\left(1 - \frac{100}{184}\right) = 310.$$

Hence if the experiment is started with 100 rats, then there will be 310 in three months time, according to the model.

Solution 3.3

(a) The proportionate growth rate has the form $r(1 - P/E)$ for a population P. Hence, working with P in billions, we have

$$0.013 = r\left(1 - \frac{5.66}{E}\right), \quad 0.006 = r\left(1 - \frac{6.36}{E}\right),$$

which lead to the pair of equations

$$\frac{0.013}{r} + \frac{5.66}{E} = 1 \quad \text{and} \quad \frac{0.006}{r} + \frac{6.36}{E} = 1.$$

Eliminating the $1/r$ term, by subtracting 6 times the first equation from 13 times the second, gives

$$\frac{13 \times 6.36 - 6 \times 5.66}{E} = 7, \quad \text{that is,} \quad E = 6.96.$$

On substituting this value for E into the first equation, we obtain

$$\frac{0.013}{r} + \frac{5.66}{6.96} = 1, \quad \text{that is,}$$
$$\frac{0.013}{r} = 1 - \frac{5.66}{6.96} = \frac{1.3}{6.96}.$$

Solving this equation gives $r = 0.0696$.

(b) The logistic recurrence relation in this case is

$$P_{n+1} - P_n = 0.0696P_n\left(1 - \frac{P_n}{6.96}\right),$$

where n is measured in years and P_n in billions. If $P_n = 6.06$ (in 2000), then the population in 2001 is given by

$$\begin{aligned} P_{n+1} &= 6.06 + 0.0696 \times 6.06\left(1 - \frac{6.06}{6.96}\right) \\ &= 6.114\,54; \end{aligned}$$

and the population in 2002 is given by

$$\begin{aligned} P_{n+2} &= P_{n+1} + 0.0696P_{n+1}\left(1 - \frac{P_{n+1}}{6.96}\right) \\ &\simeq 6.1662. \end{aligned}$$

Thus, according to the model, the world population in 2002 is predicted to be 6.17 billion.

Solution 5.1

Convergence or otherwise of a logistic recurrence sequence depends only on the value of the parameter r and, if it does converge, the limit is E. Convergence occurs for $0 < r \leq 2$ but not for $r > 2$.

(a) Since $r = 2.5 > 2$, the sequence is not convergent. (The long-term behaviour is in fact that of a 4-cycle.)

(b) With $r = 1.3 < 2$, the sequence converges, to the limit $E = 811$.

Solution 5.2

(a) We seek values of c for which the constant sequence, $x_n = c$, satisfies the given recurrence relation, $x_{n+1} = -0.2x_n + 12$. This gives $c = -0.2c + 12$, that is, $1.2c = 12$. Thus $c = 10$, so the only possible limit value is 10.

This is a linear recurrence system, so we apply the appropriate formula from Chapter A1 Section 4. This says that the closed-form solution for the linear recurrence system

$$x_0 = a, \quad x_{n+1} = rx_n + d \quad (\text{where } r \neq 1)$$

is given by

$$x_n = \left(a + \frac{d}{r-1}\right) r^n - \frac{d}{r-1}.$$

In the current case, we have $a = 5$, $r = -0.2$ and $d = 12$. Hence the closed-form solution is

$$\begin{aligned} x_n &= \left(5 + \frac{12}{-0.2-1}\right)(-0.2)^n - \frac{12}{-0.2-1} \\ &= -5(-0.2)^n + 10. \end{aligned}$$

Now $(-0.2)^n$ becomes arbitrarily small as n becomes large so, by the Constant Multiple Rule, the first term in the closed form also becomes arbitrarily small. Thus $-5(-0.2)^n + 10$ tends to 10 as n increases. In other words, the sequence x_n converges to the limit 10.

(b) We seek values of c for which the constant sequence, $x_n = c$, satisfies the given recurrence relation, $x_{n+1} = 2.4x_n - 7$. This gives $c = 2.4c - 7$, that is, $1.4c = 7$. Thus $c = 5$, so the only possible limit value is 5.

This is also a linear recurrence system, so applying the same formula from Chapter A1 Section 4 as in part (a), with $x_0 = a = 8$, $r = 2.4$ and $d = -7$, the closed-form solution is

$$\begin{aligned} x_n &= \left(8 + \frac{-7}{2.4-1}\right)(2.4)^n - \frac{-7}{2.4-1} \\ &= 3(2.4)^n + 5. \end{aligned}$$

Now $(2.4)^n$ becomes arbitrarily large as n becomes large, so the sequence x_n does not converge.

(c) This is the same recurrence relation as in part (b). The starting value $P_0 = 5$ gives the constant sequence $P_n = 5$, for all n. So the sequence converges to 5.

Solution 5.3

(a) As n becomes large, 2^n becomes large and hence $1 - 2^n$ become large (and negative). The sequence $a_n = 1 - 2^n$ does not converge.

(b) As n becomes large, n^2 becomes large, so $1/n^2$ converges to 0 (by the Reciprocal Rule). Hence $a_n = 1 + 1/n^2$ converges to the limit 1.

(c) As n becomes large, 2^n becomes large and hence also $1 + 2^n$ becomes large. By the Reciprocal Rule, $a_n = \dfrac{1}{1+2^n}$ converges to the limit 0.

(d) As n becomes large, $1/n$ becomes arbitrarily close to 0 (by the Reciprocal Rule), but n becomes large. Hence $a_n = n - 1/n$ becomes large, and so the sequence a_n is not convergent.

(e) The term $(-0.9)^n$ will become arbitrarily small as n becomes large. This is also true of $a_n = 6(-0.9)^n$, by the Constant Multiple Rule. Hence the sequence a_n converges to the limit 0.

(f) On dividing the top and bottom of the given expression for a_n by n, we have

$$a_n = \frac{2 - 3/n}{7/n - 5}.$$

Now $3/n$ and $7/n$ both converge to 0 for large n, so that $2 - 3/n$ converges to 2 and $7/n - 5$ converges to -5. Thus the sequence a_n converges to the limit $-\frac{2}{5}$.

(g) On dividing the top and bottom of the given expression for a_n by n^2, we have

$$a_n = \frac{1 + 1/n^2}{3 - 5/n^2}.$$

Now $1/n^2$ and $5/n^2$ both converge to 0 for large n, so that $1 + 1/n^2$ converges to 1 and $3 - 5/n^2$ converges to 3. Thus the sequence a_n converges to the limit $\frac{1}{3}$.

(h) On dividing the top and bottom of the given expression for a_n by n, we have

$$a_n = \frac{n + 1/n}{1 + 1/n}.$$

Now $1/n$ converges to 0 for large n, so that $1 + 1/n$ converges to 1. But the n in the term $n + 1/n$ becomes large, so that a_n becomes large, and so the sequence is not convergent.

(i) On dividing the top and bottom of the given expression for a_n by 3^n, we have

$$a_n = \frac{1}{2/3^n + 1}.$$

As n becomes large, 3^n becomes large and hence $1/3^n$ converges to 0 (by the Reciprocal Rule). Then $2/3^n$ converges to 0 (by the Constant Multiple Rule) and $2/3^n + 1$ converges to 1. Hence a_n converges to the limit $1/1 = 1$.

Solution 5.4

(a) This is an infinite geometric series, with $a = 1$ and $r = \frac{2}{3}$. Hence, the sum is

$$\sum_{i=0}^{\infty} (\tfrac{2}{3})^i = \frac{1}{1 - \frac{2}{3}} = 3.$$

(b) This is an infinite geometric series, with $a = \frac{3}{4}$ and $r = -\frac{3}{4}$. Hence, the sum is

$$\sum_{i=0}^{\infty} \tfrac{3}{4}(-\tfrac{3}{4})^i = \frac{\frac{3}{4}}{1 - (-\frac{3}{4})} = \tfrac{3}{7}.$$

(c) This is an infinite geometric series, with $a = (\frac{1}{2})^2 = \frac{1}{4}$ and $r = \frac{1}{2}$. Hence, the sum is

$$\sum_{i=0}^{\infty} \tfrac{1}{4}(\tfrac{1}{2})^i = \frac{\frac{1}{4}}{1 - \frac{1}{2}} = \tfrac{1}{2}.$$

Solution 5.5

(a) Put $s = 0.222\,222\ldots$. The repeating group, '2', is 1 digit long, so multiply s by 10, to obtain

$$10s = 2.222\,222\ldots = 2 + s.$$

Hence we have $s = \frac{2}{9}$.

(b) Put $s = 0.416\,241\,624\,162\ldots$. The repeating group, '4162', is 4 digits long, so multiply s by 10^4, to obtain

$$10\,000s = 4162.416\,241\,624\ldots = 4162 + s.$$

Hence we have $s = \frac{4162}{9999}$.

(c) Note that

$$0.317\,171\,717\ldots = 0.3 + \tfrac{1}{10} \times 0.171\,717\ldots.$$

Put $s = 0.171\,717\ldots$. The repeating group, '17', is 2 digits long, so multiply s by 10^2, to obtain

$$100s = 17.171\,717\ldots = 17 + s.$$

Hence we have $s = \frac{17}{99}$. Then

$$\begin{aligned} 0.317\,171\,717\ldots &= 0.3 + \tfrac{1}{10}s \\ &= \frac{3}{10} + \frac{17}{990} \\ &= \frac{314}{990} = \frac{157}{495}. \end{aligned}$$

(d) Put $s = 0.198\,019\,801\,980\ldots$. The repeating group, '1980', is 4 digits long, so multiply s by 10^4, to obtain

$$10\,000s = 1980.198\,019\,801\ldots = 1980 + s.$$

Hence we have

$$s = \frac{1980}{9999} = \frac{220}{1111} = \frac{20}{101}.$$

Solutions for Chapter B2

Solution 1.1

(a) An input of 5 litres at node A produces an output of $0.1 \times 5 = 0.5$ litres at node V.

(b) Inputting 2 litres of water at node A, 1 litre at node B and 2 litres at node C gives respectively 0.9×2, 0.5×1 and 0.25×2 litres output at node U. The total output at node U is thus

$$(0.9 \times 2) + (0.5 \times 1) + (0.25 \times 2) = 2.8 \text{ litres}.$$

Similarly, the output at node V is

$$(0.1 \times 2) + (0.5 \times 1) + (0.75 \times 2) = 2.2 \text{ litres}.$$

(c) The output at U is $0.9x + 0.5y + 0.25z$ litres. The output at V is $0.1x + 0.5y + 0.75z$ litres.

(d) The input vector is $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

(e) The matrix representing the network must have three columns (one for each input node) and two rows (one for each output node). The matrix is

$$\begin{pmatrix} 0.9 & 0.5 & 0.25 \\ 0.1 & 0.5 & 0.75 \end{pmatrix}.$$

Solution 1.2

(a) The label for each pipe starting from node A represents the amount of water in litres flowing out of the other end of the pipe when 1 litre of water is input at A. If 1 litre of water is input at A, then the total amount of water emerging from the nodes U, V and W must equal 1 litre. As 0.3 litres emerges from U and 0.2 litres from W, the amount which should emerge from V is $1 - (0.2 + 0.3) = 0.5$ litres. Thus the missing number for the pipe from A to V is 0.5.

Similarly, the missing number for the pipe from B to U is $1 - (0.15 + 0.5) = 0.35$.

(b) The matrix is $\begin{pmatrix} 0.3 & 0.35 \\ 0.5 & 0.15 \\ 0.2 & 0.5 \end{pmatrix}$.

(c) The input vector is $\begin{pmatrix} x \\ y \end{pmatrix}$, and

the output vector is $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$.

The relationship between these two vectors is represented by the matrix equation

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0.3 & 0.35 \\ 0.5 & 0.15 \\ 0.2 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution 1.3

(a) (i) The output vector is

$$\begin{pmatrix} 0.1 & 0.3 & 0.5 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.7 & 0.1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} (0.1\times 5) + (0.3\times 0) + (0.5\times 4) \\ (0.8\times 5) + (0\times 0) + (0.4\times 4) \\ (0.1\times 5) + (0.7\times 0) + (0.1\times 4) \end{pmatrix}$$

$$= \begin{pmatrix} 0.5+0+2 \\ 4+0+1.6 \\ 0.5+0+0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 2.5 \\ 5.6 \\ 0.9 \end{pmatrix}.$$

(ii) The output vector is

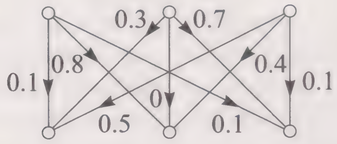
$$\begin{pmatrix} 0.1 & 0.3 & 0.5 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.7 & 0.1 \end{pmatrix} \begin{pmatrix} 100 \\ 20 \\ 60 \end{pmatrix}$$

$$= \begin{pmatrix} (0.1\times 100) + (0.3\times 20) + (0.5\times 60) \\ (0.8\times 100) + (0\times 20) + (0.4\times 60) \\ (0.1\times 100) + (0.7\times 20) + (0.1\times 60) \end{pmatrix}$$

$$= \begin{pmatrix} 10+6+30 \\ 80+0+24 \\ 10+14+6 \end{pmatrix}$$

$$= \begin{pmatrix} 46 \\ 104 \\ 30 \end{pmatrix}.$$

(b) The network diagram is shown below.



Solution 2.1

(a) $$\begin{pmatrix} 7 & 10 & -3 \\ -4 & 7 & 11 \end{pmatrix} + \begin{pmatrix} 14 & 10 & 5 \\ 4 & -20 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 20 & 2 \\ 0 & -13 & 5 \end{pmatrix}$$

(b) $$\begin{pmatrix} 7 & 10 & -3 \\ -4 & 7 & 11 \end{pmatrix} - \begin{pmatrix} 14 & 10 & 5 \\ 4 & -20 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & 0 & -8 \\ -8 & 27 & 17 \end{pmatrix}$$

(c) $$-4\begin{pmatrix} 8 & \frac{1}{2} \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -32 & -2 \\ 8 & -4 \end{pmatrix}$$

(d) $$\tfrac{2}{5}\begin{pmatrix} 7 & -15 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{14}{5} & -6 \\ -\frac{4}{5} & \frac{8}{5} \end{pmatrix}$$

(e) $$\begin{pmatrix} 3 & -1 & -2 \\ -8 & 1 & 6 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & 0 \\ 6 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -19 & 10 \\ 48 & -28 \end{pmatrix}$$

(f) $$\begin{pmatrix} 6 & 5 & 0 & 2 \\ -1 & 0 & 9 & -2 \\ 1 & 6 & 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 51 \\ -29 \\ 43 \end{pmatrix}$$

Solution 2.2

(a) (i) $a_{23} = 1$

(ii) $d_{21} = -6$

(iii) $f_{12} = -5$

(b) The sums which can be formed are $\mathbf{A}+\mathbf{F}$, $\mathbf{B}+\mathbf{B}$, $\mathbf{B}+\mathbf{E}$ and $\mathbf{F}-\mathbf{A}$.

$$\mathbf{A}+\mathbf{F} = \begin{pmatrix} -4 & -6 & 8 \\ -1 & 8 & -2 \end{pmatrix}$$

$$\mathbf{B}+\mathbf{B} = \begin{pmatrix} -4 & 8 \\ 10 & -6 \end{pmatrix}$$

$$\mathbf{B}+\mathbf{E} = \begin{pmatrix} 4 & 4 \\ 3 & -2 \end{pmatrix}$$

$$\mathbf{F}-\mathbf{A} = \begin{pmatrix} -8 & -4 & -2 \\ -1 & 2 & -4 \end{pmatrix}$$

(c) The products which can be formed are $\mathbf{BF}$, $\mathbf{BE}$, $\mathbf{EB}$, $\mathbf{FD}$, $\mathbf{DA}$, $\mathbf{B}^3$, $(\mathbf{CD})\mathbf{E}$ and $\mathbf{C}(\mathbf{DE})$.

$$\mathbf{BF} = \begin{pmatrix} 8 & 30 & -18 \\ -27 & -40 & 24 \end{pmatrix}$$

$$\mathbf{BE} = \begin{pmatrix} -20 & 4 \\ 36 & -3 \end{pmatrix}$$

$$\mathbf{EB} = \begin{pmatrix} -12 & 24 \\ 9 & -11 \end{pmatrix}$$

$$\mathbf{FD} = \begin{pmatrix} 15 & -9 \\ -43 & -5 \end{pmatrix}$$

$$\mathbf{DA} = \begin{pmatrix} 8 & 2 & 22 \\ -12 & 6 & -30 \\ 6 & 0 & 16 \end{pmatrix}$$

$$\begin{aligned} \mathbf{B}^3 &= \mathbf{BB}^2 \\ &= \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 24 & -20 \\ -25 & 29 \end{pmatrix} \\ &= \begin{pmatrix} -148 & 156 \\ 195 & -187 \end{pmatrix} \end{aligned}$$

$$(\mathbf{CD})\mathbf{E} = \begin{pmatrix} 53 & 11 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 296 & 11 \end{pmatrix}$$

$$\mathbf{C}(\mathbf{DE}) = \begin{pmatrix} 2 & -4 & 7 \end{pmatrix} \begin{pmatrix} 20 & 2 \\ -36 & 0 \\ 16 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 296 & 11 \end{pmatrix}$$

Solution 2.3

(a) We obtain

$$\mathbf{AC} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix},$$
$$\mathbf{BC} = \begin{pmatrix} 4 & 2 \\ 8 & -7 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 34 \\ 13 \end{pmatrix},$$

so that

$$\mathbf{AC} + \mathbf{BC} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 34 \\ 13 \end{pmatrix} = \begin{pmatrix} 37 \\ 19 \end{pmatrix}.$$

Also we have

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 8 & -7 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 9 & -7 \end{pmatrix},$$

so that

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \begin{pmatrix} 2 & 5 \\ 9 & -7 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 37 \\ 19 \end{pmatrix}$$
$$= \mathbf{AC} + \mathbf{BC},$$

as required.

(b) We obtain

$$\mathbf{AC} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$
$$\mathbf{BC} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} px + qy \\ rx + sy \end{pmatrix},$$

so that

$$\mathbf{AC} + \mathbf{BC} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} + \begin{pmatrix} px + qy \\ rx + sy \end{pmatrix}$$
$$= \begin{pmatrix} ax + by + px + qy \\ cx + dy + rx + sy \end{pmatrix}.$$

Also we have

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$
$$= \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix},$$

so that

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} (a+p)x + (b+q)y \\ (c+r)x + (d+s)y \end{pmatrix}$$
$$= \begin{pmatrix} ax + px + by + qy \\ cx + rx + dy + sy \end{pmatrix}$$
$$= \begin{pmatrix} ax + by + px + qy \\ cx + dy + rx + sy \end{pmatrix}$$
$$= \mathbf{AC} + \mathbf{BC},$$

as required.

Solution 3.1

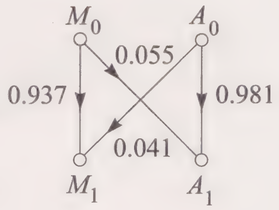
(a) Following the reasoning used for the UK population in Chapter B2 Subsection 3.1, the interpretation of the 'flow' down each pipe in the network diagram is summarised as follows.

From	To	Interpretation
M_0	M_1	Proportion of minors who neither die nor reach age 18 in year 0, plus proportionate birth rate for minors
M_0	A_1	Proportion of surviving minors who reach age 18 in year 0
A_0	M_1	Proportionate birth rate for adults in year 0
A_0	A_1	Proportion of adults who do not die in year 0

We can then calculate the numbers labelling pipes as follows, giving answers to 3 d.p.

Pipe		
From	To	Label
M_0	M_1	$\frac{17}{18}(1 - 0.011) + 0.003 = 0.937$
M_0	A_1	$\frac{1}{18}(1 - 0.011) = 0.055$
A_0	M_1	0.041
A_0	A_1	$1 - 0.019 = 0.981$

These labels appear on the network diagram shown below.



(b) The corresponding matrix model is

$$\begin{pmatrix} M_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} 0.937 & 0.041 \\ 0.055 & 0.981 \end{pmatrix} \begin{pmatrix} M_n \\ A_n \end{pmatrix}$$
$$(n = 0, 1, 2, \ldots).$$

(c) The total population predicted by the model for 1 January 2002 is $M_2 + A_2$. Using the answer to part (b), we have

$$\begin{pmatrix} M_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} 0.937 & 0.041 \\ 0.055 & 0.981 \end{pmatrix} \begin{pmatrix} M_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 0.937 & 0.041 \\ 0.055 & 0.981 \end{pmatrix} \begin{pmatrix} 3.2 \\ 13.5 \end{pmatrix} \simeq \begin{pmatrix} 3.552 \\ 13.420 \end{pmatrix};$$

and then

$$\begin{pmatrix} M_2 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0.937 & 0.041 \\ 0.055 & 0.981 \end{pmatrix} \begin{pmatrix} M_1 \\ A_1 \end{pmatrix} \simeq \begin{pmatrix} 0.937 & 0.041 \\ 0.055 & 0.981 \end{pmatrix} \begin{pmatrix} 3.552 \\ 13.420 \end{pmatrix} \simeq \begin{pmatrix} 3.878 \\ 13.360 \end{pmatrix}.$$

Thus the predicted total population for 1 January 2002 is

$$3.878 + 13.360 \simeq 17.2 \text{ (million)}.$$

Solution 5.1

(a) Since

$$\det \mathbf{A} = 7 \times 2 - 2 \times 6 = 14 - 12 = 2,$$

we have

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ -6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & \frac{7}{2} \end{pmatrix}.$$

(b) Since

$$\det \mathbf{B} = 3 \times 2 - (-6) \times (-1) = 6 - 6 = 0,$$

the matrix $\mathbf{B}$ is non-invertible, that is, $\mathbf{B}^{-1}$ does not exist.

(c) Since

$$\det \mathbf{C} = \tfrac{1}{3} \times (-\tfrac{1}{2}) - \tfrac{1}{5} \times 5 = -\tfrac{1}{6} - 1 = -\tfrac{7}{6},$$

we have

$$\mathbf{C}^{-1} = \frac{1}{-\frac{7}{6}}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{5} \\ -5 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & \frac{6}{35} \\ \frac{30}{7} & -\frac{2}{7} \end{pmatrix}.$$

Solution 5.2

(a) The determinant is

$$\det \mathbf{A} = a \times a - b \times (-b) = a^2 + b^2.$$

(b) The matrix $\mathbf{A}$ has an inverse provided that $\det \mathbf{A} \neq 0$. As a^2 and b^2 are always greater than or equal to 0, $a^2 + b^2$ is 0 only when $a = b = 0$. Thus $\mathbf{A}$ has an inverse for all values of a and b except $a = b = 0$, and the inverse is

$$\mathbf{A}^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \dfrac{a}{a^2+b^2} & \dfrac{-b}{a^2+b^2} \\ \dfrac{b}{a^2+b^2} & \dfrac{a}{a^2+b^2} \end{pmatrix}.$$

Solution 5.3

(a) The matrix form of the equations is

$$\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

The inverse matrix of $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$ is

$$\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix},$$

so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + (-1) \times 5 \\ -3 \times 2 + 4 \times 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 14 \end{pmatrix}.$$

The solution is $x = -3$, $y = 14$.

(b) The matrix form of the equations is

$$\begin{pmatrix} -2 & 5 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}.$$

The inverse matrix of $\mathbf{A} = \begin{pmatrix} -2 & 5 \\ 3 & 7 \end{pmatrix}$ is

$$\mathbf{A}^{-1} = \frac{1}{-29}\begin{pmatrix} 7 & -5 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{7}{29} & \frac{5}{29} \\ \frac{3}{29} & \frac{2}{29} \end{pmatrix},$$

so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{7}{29} & \frac{5}{29} \\ \frac{3}{29} & \frac{2}{29} \end{pmatrix}\begin{pmatrix} 8 \\ -2 \end{pmatrix} = \begin{pmatrix} -\frac{7}{29} \times 8 + \frac{5}{29} \times (-2) \\ \frac{3}{29} \times 8 + \frac{2}{29} \times (-2) \end{pmatrix} = \begin{pmatrix} -\frac{56}{29} - \frac{10}{29} \\ \frac{24}{29} - \frac{4}{29} \end{pmatrix} = \begin{pmatrix} -\frac{66}{29} \\ \frac{20}{29} \end{pmatrix}.$$

The solution is $x = -\frac{66}{29}$, $y = \frac{20}{29}$.

(c) The matrix form of the equations is

$$\begin{pmatrix} \frac{1}{3} & 3 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}.$$

The inverse matrix of $\mathbf{A} = \begin{pmatrix} \frac{1}{3} & 3 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix}$ is

$$\mathbf{A}^{-1} = \frac{1}{-2} \begin{pmatrix} -\frac{3}{2} & -3 \\ -\frac{1}{2} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{3}{2} \\ \frac{1}{4} & -\frac{1}{6} \end{pmatrix},$$

so

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{3}{4} & \frac{3}{2} \\ \frac{1}{4} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} -8 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4} \times (-8) + \frac{3}{2} \times 6 \\ \frac{1}{4} \times (-8) + (-\frac{1}{6}) \times 6 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -3 \end{pmatrix}. \end{aligned}$$

The solution is $x = 3$, $y = -3$.

Solution 5.4

(a) The matrix form of the equations is

$$\begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$7 \times 1 - 3 \times (-3) = 16.$$

This is non-zero, so the coefficient matrix is invertible and the equations have a unique solution.

(b) The matrix form of the equations is

$$\begin{pmatrix} -4 & 6 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$(-4) \times (-9) - 6 \times 6 = 0.$$

This is zero, so the coefficient matrix is non-invertible and the equations do not have a unique solution.

Solutions for Chapter B3

Solution 1.1

(a) The column forms for the three vectors are

$$\mathbf{a} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -7 \\ 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

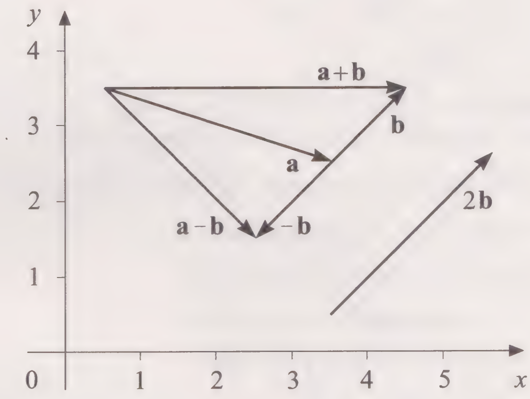
(b) The component form of $2\mathbf{a} - 3\mathbf{b} + 4\mathbf{c}$ is

$$\begin{aligned} &2(4\mathbf{i} - 2\mathbf{j}) - 3(-7\mathbf{i}) + 4(\mathbf{i} + 3\mathbf{j}) \\ &= (2 \times 4 - 3 \times (-7) + 4)\mathbf{i} \\ &\quad + (2 \times (-2) + 0 + 4 \times 3)\mathbf{j} \\ &= 33\mathbf{i} + 8\mathbf{j}, \end{aligned}$$

which has **i**-component 33 and **j**-component 8.

Solution 1.2

A labelled arrow to represent each of the vectors appears in the diagram below. The Triangle Rule is used to obtain the arrows for $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.

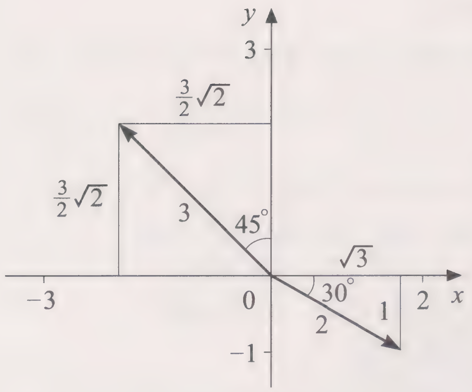


Solution 2.1

Applying the equation $\mathbf{a} = |\mathbf{a}| \cos\theta\, \mathbf{i} + |\mathbf{a}| \sin\theta\, \mathbf{j}$, where $|\mathbf{a}| = 12$ and $\theta = 100°$, we obtain the component form

$$\begin{aligned} \mathbf{a} &= 12\cos(100°)\mathbf{i} + 12\sin(100°)\mathbf{j} \\ &= -2.08\mathbf{i} + 11.82\mathbf{j} \quad \text{(to 2 d.p.)}. \end{aligned}$$

Solution 2.2



(a) Here $|\mathbf{a}| = 2$ and $\theta = -30°$. Using the trigonometric identities (from Chapter A2)

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta,$$

we obtain the component form

$$\begin{aligned}\mathbf{a} &= 2\cos(-30°)\mathbf{i} + 2\sin(-30°)\mathbf{j}\\ &= 2\cos(30°)\mathbf{i} - 2\sin(30°)\mathbf{j}\\ &= 2 \times \tfrac{1}{2}\sqrt{3}\mathbf{i} - 2 \times \tfrac{1}{2}\mathbf{j}\\ &= \sqrt{3}\mathbf{i} - \mathbf{j}.\end{aligned}$$

(b) Here $|\mathbf{a}| = 3$ and $\theta = 135°$. Using the trigonometric identities (from Chapter A2)

$$\cos(180°-\theta) = -\cos\theta, \quad \sin(180°-\theta) = \sin\theta,$$

we obtain the component form

$$\begin{aligned}\mathbf{a} &= 3\cos(135°)\mathbf{i} + 3\sin(135°)\mathbf{j}\\ &= -3\cos(45°)\mathbf{i} + 3\sin(45°)\mathbf{j}\\ &= -3 \times \tfrac{1}{2}\sqrt{2}\mathbf{i} + 3 \times \tfrac{1}{2}\sqrt{2}\mathbf{j}\\ &= -\tfrac{3}{2}\sqrt{2}\mathbf{i} + \tfrac{3}{2}\sqrt{2}\mathbf{j}.\end{aligned}$$

Solution 2.3

We follow the strategy in Chapter B3 Subsection 2.1.

The magnitude of the vector $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$ is

$$|\mathbf{a}| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5.$$

Since the components are $a_1 = 4$, $a_2 = -3$, we have

$$\phi = \arctan(|-3/4|) = \arctan\tfrac{3}{4} \simeq 36.9°.$$

Also $(4, -3)$ lies in the fourth quadrant, so that the direction of $\mathbf{a}$ is $\theta = -\phi \simeq -36.9°$.

The magnitude of $\mathbf{b} = 7\mathbf{j}$ is 7 and the direction is $\theta = 90°$.

The magnitude of the vector $\mathbf{c} = 5\mathbf{i} + 2\mathbf{j}$ is

$$|\mathbf{c}| = \sqrt{5^2 + 2^2} = \sqrt{29} \simeq 5.4.$$

Since the components are $c_1 = 5$, $c_2 = 2$, we have

$$\phi = \arctan(|2/5|) = \arctan\tfrac{2}{5} \simeq 21.8°.$$

Also $(5, 2)$ lies in the first quadrant, so that the direction of $\mathbf{c}$ is $\theta = \phi \simeq 21.8°$.

The difference of the vectors $\mathbf{a}$ and $\mathbf{c}$ is

$$\mathbf{d} = \mathbf{a} - \mathbf{c} = (4-5)\mathbf{i} + (-3-2)\mathbf{j} = -\mathbf{i} - 5\mathbf{j}.$$

The magnitude of this vector is

$$|\mathbf{d}| = \sqrt{(-1)^2 + (-5)^2} = \sqrt{26} \simeq 5.1.$$

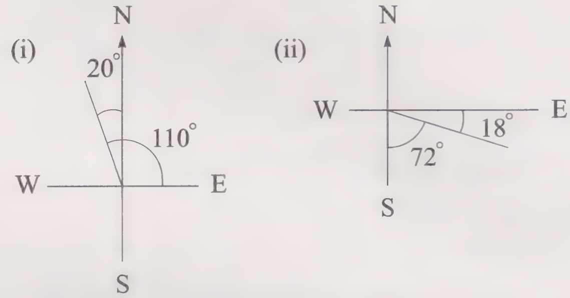
Since the components are $d_1 = -1$, $d_2 = -5$, we have

$$\phi = \arctan(|-5/(-1)|) = \arctan 5 \simeq 78.7°.$$

Also $(-1, -5)$ lies in the third quadrant, so that the direction of $\mathbf{d}$ is $\theta = -(180° - \phi) \simeq -101.3°$.

Solution 2.4

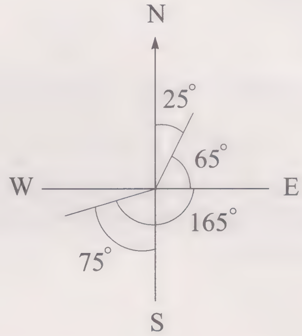
(a) The directions are shown below.



(i) The direction is $\theta = 90° + 20° = 110°$.

(ii) The direction is $\theta = -90° + 72° = -18°$.

(b) The directions are shown below.



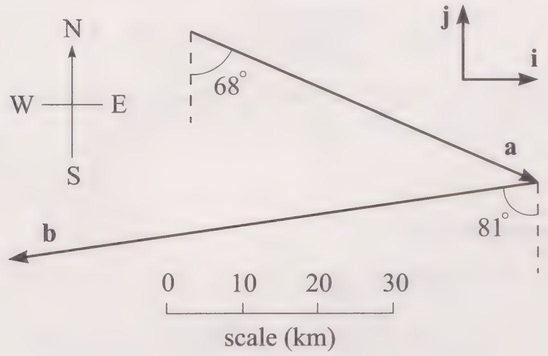
(i) The bearing is N 25°E.

(ii) The bearing is S 75°W.

Solution 2.5

(a) The displacement from Stirling to Edinburgh is the opposite of the given displacement from Edinburgh to Stirling, so it is 50 km at S 68°E.

(b) Let $\mathbf{i}$ be 1 km East and $\mathbf{j}$ be 1 km North. Denote the displacement from Stirling to Edinburgh by the vector $\mathbf{a}$, and the displacement from Edinburgh to Glasgow by the vector $\mathbf{b}$. Then the required displacement from Stirling to Glasgow is represented by the vector $\mathbf{a}+\mathbf{b}$. The vectors $\mathbf{a}$ and $\mathbf{b}$ are shown below.



The vector $\mathbf{a}$ has magnitude $|\mathbf{a}| = 50$ and direction $-90° + 68° = -22°$, so its component form is

$$\begin{aligned}\mathbf{a} &= 50\cos(-22°)\mathbf{i} + 50\sin(-22°)\mathbf{j}\\ &\simeq 46.36\mathbf{i} - 18.73\mathbf{j}.\end{aligned}$$

The vector $\mathbf{b}$ has magnitude $|\mathbf{b}| = 69$ and direction $-(90° + 81°) = -171°$, so its component form is

$$\begin{aligned}\mathbf{b} &= 69\cos(-171°)\mathbf{i} + 69\sin(-171°)\mathbf{j}\\ &\simeq -68.15\mathbf{i} - 10.79\mathbf{j}.\end{aligned}$$

The resultant is

$$\begin{aligned}\mathbf{c} &= \mathbf{a}+\mathbf{b}\\ &\simeq (46.36 - 68.15)\mathbf{i} + (-18.73 - 10.79)\mathbf{j}\\ &\simeq -21.79\mathbf{i} - 29.52\mathbf{j}.\end{aligned}$$

The vector $\mathbf{c}$ has magnitude

$$|\mathbf{c}| \simeq \sqrt{(-21.79)^2 + (-29.52)^2} \simeq 36.7.$$

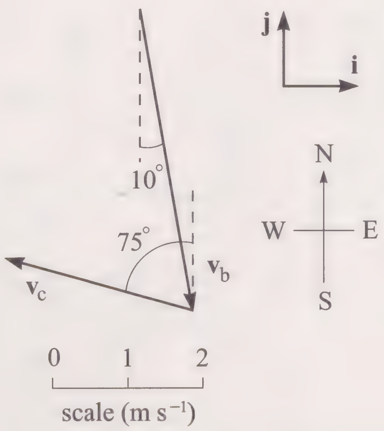
Since the components are $c_1 \simeq -21.79$, $c_2 \simeq -29.52$, we have

$$\phi \simeq \arctan(|-29.52/(-21.79)|) \simeq 53.6°.$$

Also $(-21.79, -29.52)$ lies in the third quadrant, so that the direction of $\mathbf{c}$ is $\theta = -(180° - \phi) \simeq -126.4°$. This corresponds to the bearing S 36.4°W. Hence the displacement from Stirling to Glasgow is 36.7 km at S 36.4°W.

Solution 2.6

Let $\mathbf{i}$ be $1\,\mathrm{m\,s^{-1}}$ East and $\mathbf{j}$ be $1\,\mathrm{m\,s^{-1}}$ North. Suppose that $\mathbf{v}_\mathrm{b}$ is the velocity of the boat in still water and $\mathbf{v}_\mathrm{c}$ is the velocity of the current. These vectors are shown below.



From the information given, $\mathbf{v}_\mathrm{b}$ has magnitude $|\mathbf{v}_\mathrm{b}| = 4$ and direction $\theta_\mathrm{b} = -90° + 10° = -80°$, while $\mathbf{v}_\mathrm{c}$ has magnitude $|\mathbf{v}_\mathrm{c}| = 2.5$ and direction $\theta_\mathrm{c} = 90° + 75° = 165°$.

The component forms of the vectors are therefore

$$\begin{aligned}\mathbf{v}_\mathrm{b} &= 4\cos(-80°)\mathbf{i} + 4\sin(-80°)\mathbf{j}\\ &\simeq 0.6946\mathbf{i} - 3.9392\mathbf{j},\\ \mathbf{v}_\mathrm{c} &= 2.5\cos(165°)\mathbf{i} + 2.5\sin(165°)\mathbf{j}\\ &\simeq -2.4148\mathbf{i} + 0.6470\mathbf{j}.\end{aligned}$$

The resultant velocity is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_\mathrm{b} + \mathbf{v}_\mathrm{c}\\ &\simeq (0.6946 - 2.4148)\mathbf{i} + (-3.9392 + 0.6470)\mathbf{j}\\ &\simeq -1.7202\mathbf{i} - 3.2922\mathbf{j}.\end{aligned}$$

The resultant speed of the boat is

$$|\mathbf{v}| \simeq \sqrt{(-1.7202)^2 + (-3.2922)^2} \simeq 3.7.$$

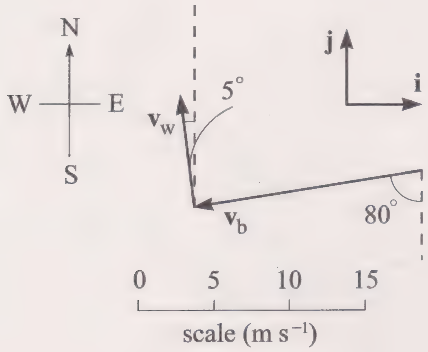
Since the components of $\mathbf{v}$ are $v_1 \simeq -1.7202$, $v_2 \simeq -3.2922$, we have

$$\begin{aligned}\phi &\simeq \arctan(|-3.2922/(-1.7202)|)\\ &\simeq \arctan(3.2922/1.7202) \simeq 62°.\end{aligned}$$

Also $(-1.7202, -3.2922)$ lies in the third quadrant, so that the direction of $\mathbf{v}$ is $\theta = -(180° - \phi) \simeq -118°$. This corresponds to the bearing S 28°W. Thus the resultant velocity of the boat is $3.7\,\mathrm{m\,s^{-1}}$ at S 28°W.

Solution 2.7

Let $\mathbf{i}$ be $1\,\mathrm{m\,s^{-1}}$ East and $\mathbf{j}$ be $1\,\mathrm{m\,s^{-1}}$ North. Suppose that $\mathbf{v}_\mathrm{b}$ is the velocity of the bird in still air and $\mathbf{v}_\mathrm{w}$ is the velocity of the wind. These vectors are shown below.



From the information given, $\mathbf{v}_\mathrm{b}$ has magnitude $|\mathbf{v}_\mathrm{b}| = 15$ and direction $\theta_\mathrm{b} = -(90° + 80°) = -170°$.

The wind comes from S 5°E and hence towards N 5°W, for which the direction is $90° + 5° = 95°$. Hence the vector $\mathbf{v}_\mathrm{w}$ has magnitude $|\mathbf{v}_\mathrm{w}| = 8$ and direction $\theta_\mathrm{w} = 95°$.

The component forms of the vectors are therefore

$$\begin{aligned}\mathbf{v}_\mathrm{b} &= 15\cos(-170°)\mathbf{i} + 15\sin(-170°)\mathbf{j}\\ &\simeq -14.772\mathbf{i} - 2.605\mathbf{j},\\ \mathbf{v}_\mathrm{w} &= 8\cos(95°)\mathbf{i} + 8\sin(95°)\mathbf{j}\\ &\simeq -0.697\mathbf{i} + 7.970\mathbf{j}.\end{aligned}$$

The resultant velocity is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_\mathrm{b} + \mathbf{v}_\mathrm{w}\\ &\simeq (-14.772 - 0.697)\mathbf{i} + (-2.605 + 7.970)\mathbf{j}\\ &\simeq -15.469\mathbf{i} + 5.365\mathbf{j}.\end{aligned}$$

The resultant speed of the bird is

$$|\mathbf{v}| \simeq \sqrt{(-15.469)^2 + (5.365)^2} \simeq 16.4.$$

Since the components of $\mathbf{v}$ are $v_1 \simeq -15.469$, $v_2 \simeq 5.365$, we have

$$\phi \simeq \arctan(|5.365/(-15.469)|) \simeq 19.1°.$$

Also $(-15.469, 5.365)$ lies in the second quadrant, so that the direction of $\mathbf{v}$ is $\theta = 180° - \phi \simeq 160.9°$. This corresponds to the bearing N 70.9°W. Thus the resultant velocity of the bird (which is its velocity relative to the ground) is $16.4\,\mathrm{m\,s^{-1}}$ at N 70.9°W.

Solution 3.1

We have $b = 9$, $A = 24°$ and $C = 53°$. The third angle of the triangle is

$$B = 180° - 24° - 53° = 103°.$$

Using the Sine Rule, the remaining side lengths are

$$a = \frac{b\sin A}{\sin B} = \frac{9\sin(24°)}{\sin(103°)} = 3.8 \quad \text{(to 1 d.p.)},$$

$$c = \frac{b\sin C}{\sin B} = \frac{9\sin(53°)}{\sin(103°)} = 7.4 \quad \text{(to 1 d.p.)}.$$

Hence, in $\triangle ABC$, we have $a = 3.8$, $b = 9$, $c = 7.4$, $A = 24°$, $B = 103°$ and $C = 53°$.

Solution 3.2

(a) By the Sine Rule,

$$\sin B = \frac{b\sin C}{c} = \frac{14\sin(20°)}{7} \simeq 0.6840.$$

This gives the possible solutions

$$\begin{aligned}B &\simeq \arcsin(0.6840) \simeq 43.2°, \quad \text{or}\\ B &\simeq 180° - 43.2° = 136.8°.\end{aligned}$$

(b) Given that BC, of length a, is the longest side of the triangle, A must be the largest angle of the triangle. This rules out the possibility that $B \simeq 136.8°$, which would make B, rather than A, the largest angle. This means that $B \simeq 43.2°$, so that

$$A \simeq 180° - 20° - 43.2° = 116.8°.$$

The only further value required to solve the triangle is that of a, which is given by

$$a = \frac{c\sin A}{\sin C} \simeq \frac{7\sin(116.8°)}{\sin(20°)} \simeq 18.3.$$

Solution 3.3

Applying the Cosine Rule in the form

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

we have

$$\cos A = \frac{14^2 + 11^2 - 6^2}{2 \times 14 \times 11} \simeq 0.912\,34,$$

so

$$A \simeq \arccos(0.912\,34) \simeq 24.17°.$$

Applying the Cosine Rule in the form

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

we have

$$\cos B = \frac{6^2 + 11^2 - 14^2}{2 \times 6 \times 11} \simeq -0.295\,45,$$

so

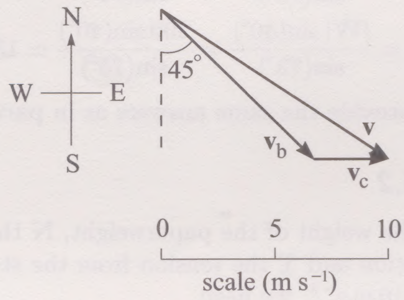
$$B \simeq \arccos(-0.295\,45) \simeq 107.18°.$$

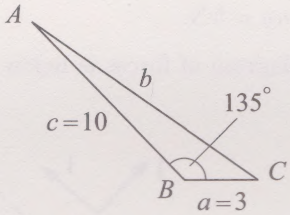
The remaining angle, C, could also be found by applying the Cosine Rule. However, it is quicker to note that

$$C = 180° - A - B$$
$$\simeq 180° - 24.17° - 107.18° = 48.65°.$$

Solution 3.4

Let $\mathbf{v}_b$ be the velocity of the boat in still water, $\mathbf{v}_c$ be the velocity of the current, and $\mathbf{v}$ be their resultant velocity. The Triangle Rule diagram for the specified situation is below, together with a separate triangle that shows the given side length and angle information.





We know that $a = 3$, $c = 10$ and $B = 180° - 45° = 135°$. We wish to find b (the magnitude of $\mathbf{v}$) and A (since the bearing for $\mathbf{v}$ is S $(45° + A)$ E).

Using the Cosine Rule in the form

$$b^2 = a^2 + c^2 - 2ac\cos B,$$

we have

$$b^2 = 3^2 + 10^2 - 2 \times 3 \times 10\cos(135°) \simeq 151.4,$$

so that $b = 12.3$ (to 1 d.p.).

Applying the Sine Rule gives

$$\sin A = \frac{a\sin B}{b} \simeq \frac{3\sin(135°)}{12.3} \simeq 0.1724.$$

Since $a < b$, and hence $A < B$, we find $A = 9.9°$ (to 1 d.p.). Hence the resultant velocity of the boat is $12.3\,\mathrm{m\,s^{-1}}$ at S 54.9°E.

Solution 4.1

Since the object remains at rest, the Equilibrium Condition applies; that is,

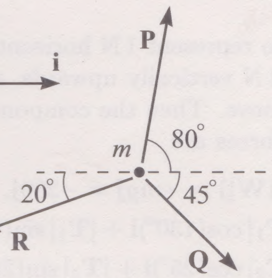
$$\mathbf{P} + \mathbf{Q} + \mathbf{R} = \mathbf{0}.$$

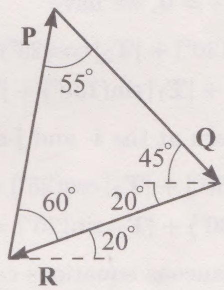
Hence we find that

$$\mathbf{Q} = -\mathbf{P} - \mathbf{R}$$
$$= -(-5\mathbf{i} + \mathbf{j}) - (3\mathbf{i} + 4\mathbf{j}) = 2\mathbf{i} - 5\mathbf{j}.$$

Solution 4.2

The force diagram and corresponding triangle of forces are below.





All of the angles in the triangle of forces can be found from the given force directions, so the Sine Rule can be applied directly, in the form

$$\frac{|\mathbf{P}|}{\sin(65°)} = \frac{|\mathbf{Q}|}{\sin(60°)} = \frac{|\mathbf{R}|}{\sin(55°)}.$$

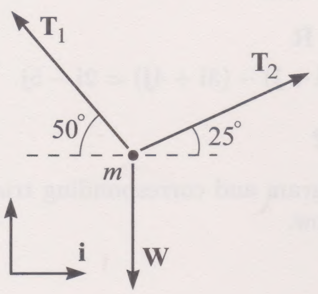
Hence the required force magnitudes are given by

$$|\mathbf{Q}| = \frac{|\mathbf{P}|\sin(60°)}{\sin(65°)} = \frac{70\sin(60°)}{\sin(65°)} \simeq 66.9\,\mathrm{N},$$

$$|\mathbf{R}| = \frac{|\mathbf{P}|\sin(55°)}{\sin(65°)} = \frac{70\sin(55°)}{\sin(65°)} \simeq 63.3\,\mathrm{N}.$$

Solution 5.1

Let $\mathbf{W}$ be the weight of the barrel, $\mathbf{T}_1$ the tension from the cord at 50° to the ceiling and $\mathbf{T}_2$ the tension from the other cord. The diagram of forces is below.



(a) Choose $\mathbf{i}$ to represent 1 N horizontally and $\mathbf{j}$ to represent 1 N vertically upwards, as in the diagram above. Then the component forms of the three forces are

$$\mathbf{W} = -|\mathbf{W}|\mathbf{j} = -mg\mathbf{j} = -200\mathbf{j},$$
$$\mathbf{T}_1 = |\mathbf{T}_1|\cos(130°)\mathbf{i} + |\mathbf{T}_1|\sin(130°)\mathbf{j},$$
$$\mathbf{T}_2 = |\mathbf{T}_2|\cos(25°)\mathbf{i} + |\mathbf{T}_2|\sin(25°)\mathbf{j}.$$

On applying the Equilibrium Condition, $\mathbf{W} + \mathbf{T}_1 + \mathbf{T}_2 = \mathbf{0}$, we have

$$(|\mathbf{T}_1|\cos(130°) + |\mathbf{T}_2|\cos(25°))\mathbf{i} + (-200 + |\mathbf{T}_1|\sin(130°) + |\mathbf{T}_2|\sin(25°))\mathbf{j} = \mathbf{0}.$$

Looking in turn at the $\mathbf{i}$- and $\mathbf{j}$-components gives

$$|\mathbf{T}_1|\cos(130°) + |\mathbf{T}_2|\cos(25°) = 0,$$
$$|\mathbf{T}_1|\sin(130°) + |\mathbf{T}_2|\sin(25°) = 200.$$

These simultaneous equations can be solved in a number of ways. For instance, from the first equation,

$$|\mathbf{T}_2| = -\frac{\cos(130°)}{\cos(25°)}|\mathbf{T}_1|.$$

Using this to substitute for $|\mathbf{T}_2|$ in the second equation gives

$$|\mathbf{T}_1|\left(\sin(130°) - \frac{\cos(130°)}{\cos(25°)}\sin(25°)\right) = 200,$$

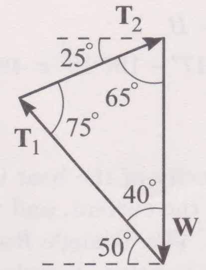
so

$$|\mathbf{T}_1| = \frac{200}{\sin(130°) - \dfrac{\cos(130°)}{\cos(25°)}\sin(25°)} \simeq 187.7\,\text{N},$$

and

$$|\mathbf{T}_2| = -\frac{\cos(130°)}{\cos(25°)}|\mathbf{T}_1| \simeq 133.1\,\text{N}.$$

(b) The triangle of forces is below.



Using the Sine Rule,

$$|\mathbf{T}_1| = \frac{|\mathbf{W}|\sin(65°)}{\sin(75°)} = \frac{200\sin(65°)}{\sin(75°)} \simeq 187.7\,\text{N},$$
$$|\mathbf{T}_2| = \frac{|\mathbf{W}|\sin(40°)}{\sin(75°)} = \frac{200\sin(40°)}{\sin(75°)} \simeq 133.1\,\text{N},$$

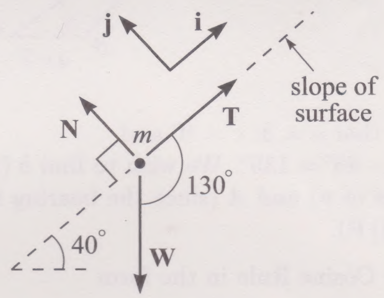
which provide the same answers as in part (a).

Solution 5.2

Let $\mathbf{W}$ be the weight of the paperweight, $\mathbf{N}$ the normal reaction and $\mathbf{T}$ the tension from the string. Taking $g = 10\,\text{m}\,\text{s}^{-2}$, we have

$$|\mathbf{W}| = mg = 5\,\text{N}.$$

(a) The diagram of forces is below.



The component forms of the three forces are

$$\mathbf{W} = |\mathbf{W}|\cos(-130°)\mathbf{i} + |\mathbf{W}|\sin(-130°)\mathbf{j}$$
$$= -5\cos(50°)\mathbf{i} - 5\sin(50°)\mathbf{j},$$
$$\mathbf{N} = |\mathbf{N}|\mathbf{j},$$
$$\mathbf{T} = |\mathbf{T}|\mathbf{i}.$$

On applying the Equilibrium Condition, $\mathbf{W} + \mathbf{N} + \mathbf{T} = \mathbf{0}$, we have

$$(-5\cos(50°) + |\mathbf{T}|)\mathbf{i} + (-5\sin(50°) + |\mathbf{N}|)\mathbf{j} = \mathbf{0},$$

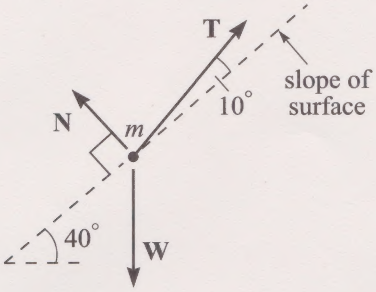
which is equivalent to

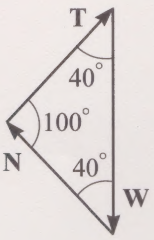
$$|\mathbf{T}| = 5\cos(50°) \quad \text{and} \quad |\mathbf{N}| = 5\sin(50°).$$

It follows that the magnitudes of the tension from the string and the normal reaction are, respectively,

$$|\mathbf{T}| \simeq 3.2\text{ N} \quad \text{and} \quad |\mathbf{N}| \simeq 3.8\text{ N}.$$

(b) The force diagram in this case and corresponding triangle of forces are below.





Applying the Sine Rule,

$$|\mathbf{T}| = \frac{|\mathbf{W}|\sin(40^\circ)}{\sin(100^\circ)} = \frac{5\sin(40^\circ)}{\sin(100^\circ)} \simeq 3.3\,\mathrm{N}.$$

(Note that the value of $|\mathbf{N}|$ is the same as that of $|\mathbf{T}|$ in this case, since the triangle of forces is isosceles.)

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